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Discrete Mathematics 308 (2008) 5284–5288

DISCRETE MATHEMATICS

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Note

On multipartite posets

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Received 30 May 2007; received in revised form 31 August 2007; accepted 3 September 2007
Available online 10 October 2007

Abstract

A poset $\mathbf{P} = (X, \preccurlyeq)$ is *m-partite* if X has a partition $X = X_1 \cup \cdots \cup X_m$ such that (1) each X_i forms an antichain in $\mathbf{P}$, and (2) $x \prec y$ implies $x \in X_i$ and $y \in X_j$ where $i < j$. In this article we derive a tight asymptotic upper bound on the order dimension of m-partite posets in terms of m and their bipartite sub-posets in a constructive and elementary way.

MSC: 06A05; 06A06; 06A07

Keywords: Linear order; Multipartite poset; Order dimension

1. Introduction

The purpose of this article is to derive an asymptotically tight upper bound for the dimension of multipartite posets in terms of their number of parts and their bipartite sub-posets. Precise definitions of terms will be given later in Section 2. This work was partly inspired by a question asked by Laubenbacher [6] which casually can be phrased as follows: "For a given collection of posets, form a new poset by stacking them together, putting one on top of the other. Is it possible to bound the order dimension of the newly formed poset in terms of the order dimension of the given posets?" Laubenbacher's motivation were posets that appeared in the following manner: When finitely many agents $A_1, \ldots, A_n$ are investigated over discrete times $t = 0, 1, \ldots, m$, one obtains a poset consisting of the $n(m+1)$ elements $A_i(t)$, where a directed edge from $A_i(t)$ down to $A_j(t+1)$ is present if, and only if, agent A_i has influenced agent A_j during the time interval from t to $t+1$. This resulting induced poset is sometimes called the *influence poset* among the agents. Here we have $m+1$ parts of the influence poset, one part $X_t = \{A_1(t), \ldots, A_n(t)\}$ for each time $t = 0, 1, \ldots, m$.

Other more classical posets can also be viewed as stacked sub-posets, one on top of the other: If $\mathscr{F}_P$ is the face lattice of an n-dimensional polytope P and $\mathscr{F}_P(i, i+1)$ is the height-2 sub-poset of $\mathscr{F}_P$ consisting of the i and $(i+1)$-dimensional faces of P, then $\mathscr{F}_P$ can be thought of being formed by stacking $\mathscr{F}_P(i, i+1)$ on top of $\mathscr{F}_P(i-1, i)$ for each $i = 0, 1, \ldots, n$. In this case the stacking appears naturally since $\mathscr{F}_P$ is a *graded poset* provided with a grading function into the nonnegative integers, that maps each face of P (i.e. each element of the poset $\mathscr{F}_P$) to its dimension. (For more on graded posets see [9,8].) Determining the order dimension of face lattices of convex polytopes is hard. Some partial yet interesting results in this direction appear in [7] and later in [1]. Of particular interest in the literature is the face lattice of the standard n-simplex when viewed as the subset lattice of $\{1, \ldots, n\}$. If we let $[n] = \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote all the k-element subsets of $[n]$, then the power set $\mathbb{P}([n])$ of all subsets of $[n]$ can be partitioned

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.09.001

into $n+1$ disjoint sets $\mathbb{P}([n]) = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \cdots \cup \binom{[n]}{n}$. For $0 \leqslant k_1 < k_2 \leqslant n$ denote the poset on $\binom{[n]}{k_1} \cup \binom{[n]}{k_2}$ induced by inclusion by $\mathbf{P}(k_1, k_2; n)$. Hence, as a poset, $\mathbb{P}([n])$ can be thought of being formed by stacking the n posets $\mathbf{P}(i, i+1; n)$ for $i \in \{0, 1, \ldots, n-1\}$ one on top of the other. Investigating the order dimension $\dim(k_1, k_2; n)$ of such sub-posets $\mathbf{P}(k_1, k_2; n)$ of $\mathbb{P}([n])$ for $1 \leqslant k_1 < k_2 \leqslant n-1$ is currently an active area of research, in particular the investigation of $\dim(1, k; n)$, the order dimension of the poset $\mathbf{P}(1, k; n)$. For a good overview of some celebrated results in this direction, we refer to [10, Chapter 7, Section 2]. Since then, relatively few values of $\dim(1, k; n)$ have been determined, but in [5,3] the exact values of $\dim(2, n-2; n)$ and $\dim(k, n-k; n)$ are given, provided that certain conditions hold for k and n. Finally, in [4] a direct method to determine $\dim(1, 2; n)$ for each n is given. Hence, the case $k = 2$ for determining $\dim(1, k; n)$ is the only case which can be considered completely solved.

In what follows we will discuss a class of posets that will include the class of graded posets and the posets obtained by such "stacking" as mentioned above in an ad hoc manner. Our methods will be constructive and combinatorially elementary. In Section 2 we introduce our notation, state our definitions in a precise manner and dispatch some basic properties. In the last Section 3 we state and prove our main result of this article.

2. Definitions and basic properties

By a *poset* $\mathbf{P}$ we will always mean an ordered tuple $\mathbf{P} = (X, \preccurlyeq)$ where $\preccurlyeq$ is a reflexive, antisymmetric and transitive binary relation on X. Unless otherwise stated X is always assumed to be a finite set. We will for the most part try to be consistent with the standard notation from [10]. In particular, if two elements $x, y \in X$ are incomparable in $\mathbf{P}$, then we write $x \| y$. By $\min(\mathbf{P})$ and $\max(\mathbf{P})$ we mean the set of minimal and maximal elements of $\mathbf{P}$, respectively. As originally defined in [2] and as stated in [10], the *order dimension* of $\mathbf{P} = (X, \preccurlyeq)$, denoted by $\dim(\mathbf{P})$, is the least number $d \in \mathbb{N}$ of linear extensions $\preccurlyeq_1, \ldots, \preccurlyeq_d$ of $\preccurlyeq$ that *realize* $\preccurlyeq$. That is, for $x, y \in X$ we have $x \preccurlyeq y$ in $\mathbf{P}$ iff $x \preccurlyeq_i y$ for all $i \in [d]$.

Recall that for $n \in \mathbb{N}$, any collection S of points in the n-dimensional Euclidean space $\mathbb{R}^n$ naturally forms a poset $(S, \preccurlyeq_E)$ by $\tilde{x} \preccurlyeq_E \tilde{y} \Leftrightarrow x_i \leqslant y_i$ for each $i \in [n]$, for any $\tilde{x} = (x_1, \ldots, x_n)$ and $\tilde{y} = (y_1, \ldots, y_n)$ from S. With this in mind we have that the order dimension $\dim(\mathbf{P})$ of a poset $\mathbf{P} = (X, \preccurlyeq)$ is the least $d \in \mathbb{N}$ such that there is an injective homomorphism $\phi : \mathbf{P} \to \mathbb{R}^d$ satisfying $x \preccurlyeq y \Leftrightarrow \phi(x) \preccurlyeq_E \phi(y)$ for all $x, y \in X$. Hence the words order *dimension*. Determining the exact value of the order dimension of a poset is a hard computational problem. Even when we restrict to height-2 posets, the problem of computing their order dimensions is NP-complete [11].

Recall that the "standard example" $\mathbf{S}_n$ from [2] and [10, p. 12] is a poset $\mathbf{S}_n = (A \cup B, \preccurlyeq)$ where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ are disjoint and $a_i \prec b_j$ if, and only if, $i \neq j$. Here S_n is a height-2 poset on $2n$ elements with order dimension of n. By adding an element c_{ij} between a_i and b_j for each $i \neq j$, so $a_i \prec c_{ij} \prec b_j$, we obtain a poset $\mathbf{P} = (A \cup C \cup B, \preccurlyeq)$ on $n(n+1)$ elements induced by the $n(n-1)$ relations $a_i \prec c_{ij} \prec b_j$. Clearly, the sub-poset induced by $A \cup B$ is the standard example, so $\dim(\mathbf{P}) \geqslant n$. However, $\mathbf{P}$ is obtained by stacking the sub-poset induced by $B \cup C$ on top of the one induced by $A \cup C$, each of which has the order dimension 2. From this we obtain the following trivial but noteworthy observation.

Observation 2.1. *There is no function* $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that* $\dim(\mathbf{P}) \leqslant f(\dim(\mathbf{P}_1), \dim(\mathbf{P}_2))$ *holds in general for all posets* $\mathbf{P}$, *which are induced by two sub-posets* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *with* $\min(\mathbf{P}_1) = \max(\mathbf{P}_2)$.

Although this answers the initial motivating question of Laubenbacher from Section 1 in the negative, it does prompt us to bound the order dimension in terms of other sub-posets.

The following lemma is a direct consequence of the interpolation property for posets and the fact that each poset has a linear extension.

Lemma 2.2. *Let* $\mathbf{P}$ *be a poset and* $\mathbf{P}'$ *be an induced sub-poset of* $\mathbf{P}$. *Then any linear extension* $\mathbf{L}'$ *of* $\mathbf{P}'$ *can be extended to a linear extension* $\mathbf{L}$ *of* $\mathbf{P}$.

Recall that a *bipartite* poset is an ordered triple $\mathbf{P} = (X, Y; \preccurlyeq)$ where X and Y are disjoint and $x \prec y$ implies that $x \in X$ and $y \in Y$. This can be generalized.

Definition 2.3. Let $m \geqslant 2$ be an integer and $X_1, \ldots, X_m$ be disjoint nonempty sets. We call $\mathbf{P} = (X_1, \ldots, X_m; \preccurlyeq)$ an *m-partite poset* if $\preccurlyeq$ is a partial order on $X = X_1 \cup \cdots \cup X_m$ such that (1) each X_i forms an antichain w.r.t. $\preccurlyeq$ and (2)

$x \prec y$ implies $x \in X_i$ and $y \in X_j$ where $i, j \in [m]$ and $i < j$. If $\mathbf{P}$ is m-partite for some m, then $\mathbf{P}$ is a *multipartite poset*.

Clearly, each m-partite poset $\mathbf{P}$ yields its underlying poset $\mathbf{P}^\circ = (X_1 \cup \cdots \cup X_m, \preccurlyeq)$ by ignoring the partition. The order dimension of $\mathbf{P}$ is then defined to be that of $\mathbf{P}^\circ$.

Remark. Note that an m-partite poset $(X_1, \ldots, X_m; \preccurlyeq)$ may contain covering relations $x \preccurlyeq y$ that are *not* between consecutive levels X_i, X_{i+1}. However, each poset $\mathbf{P} = (X, \preccurlyeq)$ has a partition $X = X_1 \cup \cdots \cup X_m$ such that (1) $(X_1, \ldots, X_m; \preccurlyeq)$ is a multipartite poset with $\mathbf{P}$ as its underlying poset and (2) all the cover relations are between consecutive levels X_i, X_{i+1}. In fact, letting $X_1 = \min(\mathbf{P})$ and $X_{i+1} = \min(\mathbf{P} \setminus (X_1 \cup \cdots \cup X_i))$ for $1 \leqslant i \leqslant m-1$, will yield such a partition.

3. Multipartite posets

Let $\mathbf{P} = (X_1, \ldots, X_m; \preccurlyeq)$ be an m-partite poset and $\mathbf{P}_{i,j}$ be the bipartite sub-poset of $\mathbf{P}$ induced by $X_i \cup X_j$ for each $i < j$ with $i, j \in [m]$. By Observation 2.1, we cannot hope to express $\dim(\mathbf{P})$ in terms of the $\dim(\mathbf{P}_{i,i+1})$'s for $i \in [m-1]$, the order dimensions of these consecutive layers in $\mathbf{P}$. More is needed.

For each $i, j \in [m]$ with $i < j$ let $d_{i,j} = \dim(\mathbf{P}_{i,j})$ and $\mathscr{L}_{i,j}$ be a collection of $d_{i,j}$ linear orders on $X_i \cup X_j$ realizing $\mathbf{P}_{i,j}$. By Lemma 2.2 there is a set $\mathscr{L}^*_{i,j}$ of $d_{i,j}$ linear orders extending $\mathbf{P}$ and each linear order in $\mathscr{L}_{i,j}$. By considering both cases of $x \| y$, where $x, y \in X_i$ for some i on one hand, and $x \in X_i$, $y \in X_j$ for some $i \neq j$ on the other, we can see that $\mathscr{R} = \bigcup_{i<j} \mathscr{L}^*_{i,j}$ realizes $\mathbf{P}$. This shows that we can bound $\dim(\mathbf{P})$ in terms of the $\dim(\mathbf{P}_{i,j})$'s. We summarize in the following.

Observation 3.1. *For a multipartite poset* $\mathbf{P} = (X_1, \ldots, X_m; \preccurlyeq)$ *we have*

$$\dim(\mathbf{P}) \leqslant \sum_{i<j} \dim(\mathbf{P}_{i,j}).$$

For an m-partite poset $\mathbf{P}$ let $B(\mathbf{P}) = \max_{i<j}\{\dim(\mathbf{P}_{i,j})\}$. Since there are $\binom{m}{2} = m(m-1)/2$ posets $\mathbf{P}_{i,j}$ we obtain

$$B(\mathbf{P}) \leqslant \dim(\mathbf{P}) \leqslant \frac{m(m-1)}{2} B(\mathbf{P}),$$

and hence for a fixed m, we have $\dim(\mathbf{P}) = \Theta(B(\mathbf{P}))$. This can be reduced by a factor of $\frac{1}{2}$ in the following theorem.

Theorem 3.2. *For a multipartite poset* $\mathbf{P} = (X_1, \ldots, X_m; \preccurlyeq)$ *we have*

$$\dim(\mathbf{P}) \leqslant \left\lfloor \frac{(m-1)(m+3)}{4} \right\rfloor B(\mathbf{P}).$$

Proof. Note that if $i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell$ are indices from $[m]$ and $\mathbf{L}_k$ is a linear extension of $\mathbf{P}_{i_k, j_k}$, then a linear extension of $\mathbf{P}$ that includes $\mathbf{L}_1 \prec \mathbf{L}_2 \prec \cdots \prec \mathbf{L}_\ell$ extends $\mathbf{P}$ and each of the $\mathbf{L}_k$. In this way we can find $2 \cdot B(\mathbf{P})$ linear orders extending $\mathbf{P}$ and each $\mathbf{L}_{i,j} \in \mathscr{L}_{i,j}$, where $i + 1 = j$. In general, for each $k \leqslant \lfloor (m+1)/2 \rfloor$ there are $k \cdot B(\mathbf{P})$ linear orders extending $\mathbf{P}$ and each $\mathbf{L}_{i,j}$, where $i + k - 1 = j$. There are however $1 + 2 + \cdots + (m - \lfloor (m+1)/2 \rfloor)$ ways of choosing a pair $i < j$ with $j - i \geqslant \lfloor (m+1)/2 \rfloor$. Therefore, the total number of linear orders extending $\mathbf{P}$ and each $\mathbf{L}_{i,j} \in \mathscr{L}_{i,j}$ for all $i < j$, will not exceed

$$\left[\left(2 + 3 + \cdots + \left\lfloor \frac{m+1}{2} \right\rfloor\right) + \left(1 + 2 + \cdots + \left(m - \left\lfloor \frac{m+1}{2} \right\rfloor\right)\right)\right] \cdot B(\mathbf{P})$$
$$= \left\lfloor \frac{(m-1)(m+3)}{4} \right\rfloor B(\mathbf{P}).$$

Hence we have the theorem. $\square$

Remark. Considering the canonical interval order $\preccurlyeq$ on $\binom{[m]}{2}$ in which $\{i_1, j_1\} \prec \{i_2, j_2\}$ iff $j_1 < i_2$, we see that all the $\lfloor (m-1)(m+3)/4 \rfloor$ 2-sets $\{i, j\} \in \binom{[m]}{2}$ with $i \in \{1, \ldots, \lceil m/2 \rceil\}$ and $j \in \lceil m/2 \rceil, \ldots, m\}$ are incomparable. This means that the total number of linear orders in the proof of Theorem 3.2, that extend $\mathbf{P}$ and each $\mathbf{L}_{i,j} \in \mathscr{L}_{i,j}$, cannot be reduced any further with the arguments presented there.

To better understand the asymptotic behavior of $\dim(\mathbf{P})$ of an m-partite poset $\mathbf{P}$, define $f(m)$ for each $m \geqslant 2$ by

$$f(m) = \sup_{\mathbf{P}} \left\{ \frac{\dim(\mathbf{P})}{B(\mathbf{P})} \right\}, \tag{1}$$

where the supremum is taken over all m-partite posets $\mathbf{P}$. By Theorem 3.2 we therefore have that $f(m) \leqslant \lfloor (m-1)(m+3)/4 \rfloor$.

For the lower bound of $f(m)$, we start with the following lemma.

Lemma 3.3. *Let $g, h, k \in \mathbb{N}$ with $h, k \geqslant 2$ and $g \leqslant \min\{h, k\}$. Let $M \subseteq [h] \times [k]$ be any matching of size g between the columns and rows of $[h] \times [k]$. For disjoint sets $X = \{x_1, \ldots, x_h\}$ and $Y = \{y_1, \ldots, y_k\}$ let $\mathbf{C}_{-g}(h, k)$ be the poset on $X \cup Y$ given by $x_i \prec y_j$ for all $(i, j) \in [h] \times [k] \backslash M$. Then* $\dim(\mathbf{C}_{-g}(h, k)) = \max\{2, g\}$.

Proof. Assume $g \geqslant 2$. By a suitable permutation we may assume that $M = \{(1, 1), \ldots, (g, g)\}$. Since the poset induced by $\{x_1, \ldots, x_g\} \cup \{y_1, \ldots, y_g\}$ is the standard example $\mathbf{S}_{2g}$ we have that $\dim(\mathbf{C}_{-g}(h, k)) \geqslant g$.

Let L_x denote the linear order $x_1 \prec x_3 \prec x_4 \prec \cdots \prec x_{h-1} \prec x_h \prec x_2$ and similarly let L_y denote $y_1 \prec y_3 \prec y_4 \prec \cdots \prec y_{k-1} \prec y_k \prec y_2$. If $i \in [h]$ then $L_x(\hat{i})$ denotes the linear order obtained from L_x by removing x_i and similarly for $L_y(\hat{j})$. For any linear order L let L^{op} denote the opposite, or reverse, linear order of L. In this case $\mathbf{C}_{-g}(h, k)$ is realized by the following g linear orders:

$$L_x(\hat{1})^{\mathrm{op}} \prec y_1 \prec x_1 \prec L_y(\hat{1}),$$

$$L_x(\hat{\ell}) \prec y_\ell \prec x_\ell \prec L_y(\hat{\ell})^{\mathrm{op}} \quad \text{for } \ell \in \{2, \ldots, g\}.$$

Hence $\dim(\mathbf{C}_{-g}(h, k)) \leqslant g$. The case $g = 1$ gives in similar fashion $\dim(\mathbf{C}_{-1}(h, k)) = 2$. $\square$

Note that $\mathbf{C}_{-g}(h, k)$ is the complete bipartite poset on X and Y except for the g relations $x_i \prec y_j$ where $(i, j) \in M$.

Theorem 3.4. *For $m \geqslant 2$ we have that $f(m)$ defined in* (1) *satisfies*

$$f(m) \geqslant \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil = \left\lfloor \frac{m^2}{4} \right\rfloor.$$

Proof. For $d, h, k \geqslant 2$ let $A = \{x_{i,j} : (i, j) \in [dh] \times [dk]\}$ and $B = \{y_{i,j} : (i, j) \in [dh] \times [dk]\}$ be two disjoint sets of d^2hk elements each. Let $\mathbf{P} = (A \cup B; \preccurlyeq)$ be given by

$$x_{i_1, j_1} \prec y_{i_2, j_2} \quad \Leftrightarrow \quad (i_1, j_1) \neq (i_2, j_2).$$

Here $\mathbf{P}$ is the standard example on $2d^2hk$ elements so $\dim(\mathbf{P}) = d^2hk$. Let $X_1, \ldots, X_h, Y_1, \ldots, Y_k$ be given by $X_p = \{x_{i,j} : (i, j) \in \{(p-1)d+1, \ldots, pd\} \times [dk]\}$ for each $p \in [h]$ and $Y_q = \{y_{i,j} : (i, j) \in [dh] \times \{(q-1)d+1, \ldots, qd\}\}$ for each $q \in [k]$. This partition of $A \cup B$ makes $\mathbf{P}$ into a $(h+k)$-partite poset $(X_1, \ldots, X_h, Y_1, \ldots, Y_k; \preccurlyeq)$. We note that each of $X_p \cup X_q$ and $Y_p \cup Y_q$ is an antichain in $\mathbf{P}$ of order dimension two. Since the sub-poset of $\mathbf{P}$ induced by $X_p \cup Y_q$ is $\mathbf{C}_{-d^2}(d^2k, d^2h)$ we have by Lemma 3.3 that $B(\mathbf{P}) = d^2$. Hence we have

$$f(h+k) \geqslant \frac{\dim(\mathbf{P})}{B(\mathbf{P})} = \frac{d^2hk}{d^2} = hk.$$

Putting $(h, k) = (n, n)$ on one hand and $(h, k) = (n, n+1)$ on the other yields a lower bound for $f(m)$ both for even and odd m. Hence, we have the theorem. $\square$

Note that the example provided in the above proof of Theorem 3.4 shows that both $\dim(\mathbf{P})$ and $B(\mathbf{P})$ can be arbitrarily large.

By Theorems 3.2 and 3.4 we have the following.

Corollary 3.5. *If $f(m)$ is the function from (1), then for all $m \geqslant 2$ we have*

$$\left\lfloor \frac{m^2}{4} \right\rfloor \leqslant f(m) \leqslant \left\lfloor \frac{(m-1)(m+3)}{4} \right\rfloor.$$

By Corollary 3.5 we have $\lim_{m\to\infty} f(m)/m^2 = \frac{1}{4}$, so the upper bound in Theorem 3.2 is asymptotically tight.

Acknowledgment

The author likes to thank Reinhard Laubenbacher for interesting discussions regarding applications of posets. Also, sincere thanks to Walter D. Morris for his helpful comments on the article.

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